Conventions for Laguerre polynomials

Árpád Lukács

Kavli Institute for Quantum Nanoscience, TU Delft, Lorentzweg 1, Delft, The Netherlands

and

Wigner Research Centre for Physics, Budapest, Hungary

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Abstract

We compare the conventions used for the generalized Laguerre polynomials used in different sources. We give a Rodrigues formula and the differential equation satisfied by the most commonly used convention (used, e.g., by Abramowitz and Stegun, by the NIST Handbook of mathematical functions and by Gradshteyn and Ryzhik. For the other conventions, we present a link to the most often used one.

Our choice of the sources is somewhat arbitrary. We have compared the notation used in some more widespread textbooks and monographs on quantum mechanics, and some handbooks and textbooks of mathematics often used by physicists.

It seems, that for Laguerre polynomials many different notation conventions are used in physics literature. When using multiple references, e.g., formulae from ones favourite quantum mechanics book, and a table of integrals or a handbook of mathematics, or implementing numerical code using library functions, one may easily run into discrepancies.

The aim of the present note is to clear up the confusion of notation, to help the translation of the formulae of one reference into the other. It has been found, that at least some of the mathematical handbooks, e.g., [1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15], and the computer algebra package Mathematica [16] uses a consistent notation, which we shall discuss first, in Sec. 1, and in the subsequent sections, relate the notation of other sources to this one.

In the present note, we consider generalized Laguerre polynomials $L_n^m(x)$, where n is an integer.

1 Most often used

The most often used conventions are used by Refs. [1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15][1] and the computer algebra package [16]. These correspond to the Rodrigues formulae,

$$\begin{aligned} L_n(x) &= \frac{1}{n!}\mathrm{e}^x\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-x}x^n)\,, \\ L_n^m(x) &= \frac{1}{n!}\mathrm{e}^x x^{-m}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-x}x^{n+m})\,. \end{aligned} \tag{1}$$

See Ref. [1], 22.11.6, Ref. [2], 18.5.5, and Ref. [3], 8.970.

[1] Ref. [9] uses $L_n^m(x)$ for Laguerre polynomials, and $L_n^{(m)}(x)$ for Laguerre functions. The latter reduce to $m!L_n^m(x)/(m+n)!$, when n is an integer.

Importantly,

$$L_n^m(x) = (-1)^m \frac{\mathrm{d}^m}{x^m} L_{n+m}(x)\,, \tag{2}$$

see, e.g., Ref. [1], 22.5.17.

Note, that $L_n(x) = L_n^0(x)$. The functions are given explicitly by the formula

$$L_n^m(x) = \sum_{k=0}^{n} (-1)^k \binom{n+m}{n-k} \frac{x^k}{k!}\,, \tag{3}$$

(Ref. [1], 22.3.9). They are orthogonal functions,

$$\int_0^\infty \mathrm{d}x \mathrm{e}^{-x} x^m L_n^m(x) L_{n'}^m(x) = \frac{(n+m)!}{n!} \delta_{nn'}\,, \tag{4}$$

(Ref. [1], 22.2.13).

These functions satisfy the differential equation

$$x L_n^{m\prime\prime}(x) + (m+1-x) L_n^{m\prime}(x) + n L_n^m(x) = 0\,, \tag{5}$$

(Ref. [1], 22.6.15), and the recurrence relation

$$(n+1) L_{n+1}^m(x) = (2n+m+1-x) L_n^m(x) - (n+m) L_{n-1}^m(x)\,, \tag{6}$$

(Ref. [1], 22.7.12). Their generating function is

$$\sum_{n=0}^{\infty} L_n^m(x) s^n = \frac{\exp\left(\frac{xs}{s-1}\right)}{(1-s)^{m+1}}\,, \tag{7}$$

(Ref. [1], 22.9.15), and

$$\sum_{n=0}^{\infty} L_n^{m-n}(x) s^n = \mathrm{e}^{-sx} (1+s)^n\,, \tag{8}$$

(Ref. [11], Sec. 5.5.2), and the relation $L_n^{-m}(x) = (-x)^m (n-m)! L_{n-m}^m(x)/n!$ holds [11].

Ref. [17] does not give a definition for $L_n^m(x)$, however, in their derivation where Laguerre polynomials are used, a Taylor expansion is performed, equivalent to the generating function relation (8), with the replacement of s by $-s$.

The relation of the Laguerre polynomials to the confluent hypergeometric functions is

$$L_n^m(x) = \binom{n+m}{n} M(-n, m+1, x)\,, \tag{9}$$

where M is Kummer's hypergeometric function ([1], 22.5.54). This formula is also used by Refs. [18, 19].

Note, that many Refs., e.g., [1, 2, 3, 4], give many more useful formulae, but for our purposes, i.e., comparison among the references, these suffice. In what follows, we shall use the notations, that we reserve the symbols L_n and L_n^m for the functions defined by Eq. (1), and use the notations $\tilde{L}_n$ and $\tilde{L}_n^m$ for functions defined in other sources (and separate different ones in different sections).

2 Blokhintsev, A. Bohm, D. Bohm, Landau and Lifshitz, Sakurai, Courant and Hilbert

In Ref. [20], the eigenfunctions of the hydrogen atom are considered. The radial functions are expressed with Laguerre polynomials $\tilde{L}_n^m(x)$, which satisfy the differential equation

$$x\tilde{L}_n^{m\prime\prime}(x) + (m+1-x)\tilde{L}_n^{m\prime}(x) + (n-m)\tilde{L}_n^m(x) = 0\,, \tag{10}$$

which already tells us that the lower index is shifted by $-m$ relative to the conventions used in Sec. 1. The Reudrigues formulae are also given, as

$$\begin{aligned} \tilde{L}_n(x) &= \mathrm{e}^x \frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-x}x^n)\,, \\ \tilde{L}_n^m(x) &= (-1)^m \frac{n!}{(n-m)!}\mathrm{e}^x x^{-m} \frac{\mathrm{d}^{n-m}}{\mathrm{d}x^{n-m}}(\mathrm{e}^{-x}x^n)\,, \end{aligned} \tag{11}$$

which can be compared with Eq. (1) to yield

$$\tilde{L}_n^m(x) = (-1)^m n! L_{n-m}^m(x)\,, \qquad L_n^m(x) = \frac{(-1)^m}{(n+m)!}\tilde{L}_{n+m}^m(x)\,. \tag{12}$$

The relation to confluent hypergeometric functions is

$$\tilde{L}_n^m(x) = (-1)^m \frac{[n!]^2}{m!(n-m)!} M(-(n-m), m+1, z)\,. \tag{13}$$

Eq. (13) can also be obtained by substituting Eq. (9) into formula (12). Note, that Ref. [20] uses the notation $F(\alpha,\beta,z)$ for the hypergeometric function $M(\alpha,\beta,z)$, but gives its differential equation and first two Taylor coefficients, which can be compared with, e.g., Eqs. 13.1.1 and 13.1.2 of Ref. [1].

Refs. [21, 22, 23, 24] gives the following formulae,

$$\tilde{L}_n^m(x) = \frac{\mathrm{d}^m}{\mathrm{d}x^m}\tilde{L}_n(x)\,, \tag{14}$$

and

$$\tilde{L}_n(x) = \mathrm{e}^x \frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-x}x^n)\,. \tag{15}$$

Note, that the latter agrees with the Rodrigues formula used by Ref. [20], and differs from the one in Sec. 1 by the lack of a normalisation $1/n!$, yieding again Eq. (12).

Ref. [25] gives the Rodrigues formula in the form

$$\tilde{L}_n^m(x) = \frac{\mathrm{d}^m}{\mathrm{d}x^m}\mathrm{e}^n \frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-x}x^n)\,, \tag{16}$$

equivalent with Eqs. (14) and (15).

3 Schiff

Ref. [26] gives the differential equation of Laguere functions the same way as Ref. [20], the normalisation, however, differs. The generating function is given as

$$\sum_{n=m}^{\infty} \tilde{L}_n^m(x) s^n = \frac{(-s)^n \exp\left(\frac{xs}{s-1}\right)}{(1-s)^{m+1}}\,, \tag{17}$$

the comparison of which with Eq. (7) yields

$$\tilde{L}_n^m(x) = (-1)^m L_{n-m}^m(x)\,, \qquad L_n^m(x) = (-1)^m L_{n+m}^m(x)\,. \tag{18}$$

Note, that this differs by the lack of a multiplier $n!$ from the definition used by Ref. [20].

Note, that the functions used in Ref. [26] satisfy

$$L_n^m(x) = \frac{\mathrm{d}^m}{\mathrm{d}x^m} L_n(x)\,. \tag{19}$$

4 Griffiths, Messiah, Merzbacher, Morse and Feshbach, Shankar, Byron and Fuller

In Ref. [27], the differential euquation is

$$x\tilde{L}_n^{m\prime\prime}(x) + (m+1-x)\tilde{L}_n^{m\prime}(x) + n\tilde{L}_n^m(x) = 0\,, \tag{20}$$

in agreement with Sec. 1. The Rodrigues formula for the Laguerre polynomials is given as

$$\begin{aligned} \tilde{L}_n(x) &= \mathrm{e}^x \frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-x}x^n)\,, \\ \tilde{L}_n^m(x) &= \frac{(n+m)!}{n!}\mathrm{e}^x x^{-m}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-x}x^{n+m})\,, \end{aligned} \tag{21}$$

which, compared with Eqs. (1), yields

$$\begin{aligned} \tilde{L}_n(x) &= n!L_n(x)\,, \qquad & L_n(x) &= \tilde{L}_n(x)/n!\,, \\ \tilde{L}_n^m(x) &= (n+m)!L_n^m(x)\,, \qquad & L_n^m(x) &= \tilde{L}_n^m(x)/(n+m)!\,. \end{aligned} \tag{22}$$

These notations are also used by Refs. [28, 29, 30, 31], who gives the Rodrigues formula (21) for $\tilde{L}_n$, and the recursion (2).

Ref. [32] gives the Rodrigues formula (21) for the Laguerre polynomial $\tilde{L}_n$, and the one for the generalised Laguerre polynomial $\tilde{L}_n^m$ with an unspecified coefficient D_n^m.

5 Baym, Fock, Smirnow, Takhtadjan

Ref. [33] gives an explicit formula for $\tilde{L}_{n+\ell}^{2\ell+1}(x)$, equivalent to $n!$ times the one in Eq. (3). In Refs. [34, 35, 36], the Rodrigues formula is given as

$$\tilde{L}_n^m(x) = \mathrm{e}^x x^{-m}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-x}x^{n+m})\,, \tag{23}$$

which differs by a factor $n!$ from Eq. (1), therefore

$$\tilde{L}_n^m(x) = n!L_n^m(x)\,, \qquad L_n^m(x) = \frac{1}{n!}\tilde{L}_n^m(x)\,. \tag{24}$$

6 Frank and von Mises

Ref. [37] defines Laguerre polynomials by orthonormalisation,

$$\int_0^\infty \mathrm{e}^{-x}x^m \tilde{L}_n^m(x)\tilde{L}_{n'}^m(x) = \delta_{nn'}\,, \tag{25}$$

which, upon comparison with Eq. (4) yields $\tilde{L}_n(x) = L_n(x)$ and

$$\tilde{L}_n^m(x) = \sqrt{\frac{n!}{(n+m)!}}L_n^m(x)\,, \qquad L_n^m(x) = \sqrt{\frac{(n+m)!}{n!}}\tilde{L}_n^m(x)\,. \tag{26}$$

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