Notes on Geometry of physics by Frankel: some harder to follow derivations

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9.3. Cartan's exterior covariant derivation

I think what makes this hard is mixing index and matrix, index and vector notations, writing out some sums and not others.

9.3a. Vector-valued forms

A vector valued form is defined as a fully anti-symmetric multi-linear mapping from a vector space to another one, e.g., in the example of the text,

$$A : TM \otimes \cdots \otimes TM \to TM\,.$$

Choosing a frame $\mathbf{e} = \{\mathbf{e}_i : i = 1, \dots, n\}$ in the target space, and basis forms σ^i in the dual of the domain, such a mapping can be expanded as

$$A = \sum_i \sum_{\underset{\to}{J}} A^i_{j_1,\dots,j_p} \mathbf{e}_i \otimes \sigma^{j_1} \wedge \cdots \wedge \sigma^{j_p}\,.$$

The same set of components belong to a lot of different mappings, e.g., to one $TM \times TM \to TM$ bilinear, one $TM \otimes TM \to TM$ linear, $TM \to TM \otimes T^*M$ linear, etc. These are identified, and often the same letter is used for them.

The example given, the one-form $\mathrm{d}\mathbf{r}$ is a vector-valued one-form, given as

$$\mathrm{d}\mathbf{r} = (\mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3)^T$$

is in a mixed notation on the RHS: component (matrix) for $\mathbf{r} = (x^1, x^2, x^3)^T$, and index-free (abstract) for the forms $\mathrm{d}x^i$.

The other example is

$$\mathrm{d}\mathbf{S} = (\mathrm{d}y \wedge \mathrm{d}z, \mathrm{d}z \wedge \mathrm{d}x, \mathrm{d}x \wedge \mathrm{d}y)^T\,,$$

which is a vector-valued 2-form, $\mathrm{d}\mathbf{S} \in \Gamma(TM \otimes T^*M \wedge T^*M)$. Its components are

$$\epsilon^i{}_{jk}\,,$$

which helps us write it as

$$\mathrm{d}\mathbf{S}(\mathbf{A}, \mathbf{B}) = g^{-1}(i_{\mathbf{B}} i_{\mathbf{A}} \mathrm{vol})\,,$$

or identifying vol as a mapping from $TM \otimes TM \otimes TM \to \mathbb{R}$ and as $TM \otimes TM \to T^*M$, $(\mathbf{A}, \mathbf{B}) \mapsto \mathrm{vol}(\mathbf{A}, \mathbf{B}, \cdot)$, we obtain $\mathrm{d}\mathbf{S}(\mathbf{A}, \mathbf{B}) = g^{-1}\mathrm{vol}(\mathbf{A}, \mathbf{B}, \cdot)$, so

$$\mathrm{d}\mathbf{S} = g^{-1}\mathrm{vol}\,.$$

9.3b. The covariant differential of a vector field

The first thing introduced is a set of connection one-forms,

$$\omega^k{}_j = \sum_r \omega^k{}_{rj}\sigma^r \,.$$

This is again mixing index and abstract notations, it has indices k and j labelling the one-forms, and they are all one-forms. One could say that on the RHS, k and j are labels, r a (summation) index.

As the covariant derivative of a vector has components (in a coordinate frame)

$$\nabla_j v^i = \partial_j v^i + \omega^i{}_{jk} v^k \,,$$

or in a general frame

$$\nabla_j v^i = \mathbf{e}_k(v^i) + \omega^i{}_{jk} v^k \,,$$

we may obtain the vector field itself,

$$\nabla_j \mathbf{v} = \left(\mathbf{e}_k(v^i) + \omega^i{}_{jk} v^k\right) \mathbf{e}_i \,,$$

or for an arbitrary vector $\mathbf{X} = X^j \mathbf{e}_j$,

$$\nabla_{\mathbf{X}} \mathbf{v} = X^j \left(\mathbf{e}_k(v^i) + \omega^i{}_{jk} v^k\right) \mathbf{e}_i = \left(\mathbf{e}_k(v^i) + \omega^i{}_{jk} v^k\right) \mathbf{e}_i \sigma^j(\mathbf{X}) \,,$$

i.e.

$$\nabla \mathbf{v} = \left(\mathbf{e}_k(v^i) + \omega^i{}_{jk} v^k\right) \mathbf{e}_i \otimes \sigma^j \,,$$

is a vector-valued one-form. In particular, we may apply this to $\mathbf{e}_j$, whose coordinates are constant,

$$\nabla \mathbf{e}_j = \sum_{i,k} \omega^k{}_{ji} \mathbf{e}_k \otimes \sigma^i \,,$$

which is a vector valued 1-form with a label j, whose coefficients, when expandig w.r.t. $\mathbf{e}_k$ are the connection 1-forms,

$$\nabla \mathbf{e}_j = \sum_k \mathbf{e}_k \otimes \omega^k{}_j \,, \quad \omega^k{}_j = \sum_i \omega^k{}_{ij} \sigma^i \,.$$

With the connection one-forms, we may write the covariant derivative of an arbitrary vector $\mathbf{v} = \mathbf{e}_j v^j$ as

$$\nabla \mathbf{v} = \nabla(\mathbf{e}_j v^j) = \sum_j \mathbf{e}_j \otimes \mathrm{d}v^j + \sum_j \nabla \mathbf{e}_j v^j = \sum_k \mathbf{e}_k \otimes \left(\mathrm{d}v^k + \sum_k \omega^k{}_j v^k\right) \,,$$

or, in mixed component and abstract notation,

$$\nabla v^i = \mathrm{d}v^i + \omega^i{}_k v^k \,.$$

9.3c. Cartan's structural equations

In this section, the notational difficulty is a new notation. The tensor product and the wedge (exterior) products are extended to tensor-matrix and wedge-matrix products, i.e., in eq. (9.29) and above

$$\nabla \mathbf{e} = \mathbf{e} \otimes \omega$$

is meant in such a way: $\mathbf{e} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ is a row vector of vector fields, and $\omega = \{\omega^j{}_k\}$ is a matrix of one-forms. In the equation above, there is a "tensor-matrix" product,

$$\nabla \mathbf{e} = \nabla(\mathbf{e}_1, \ldots, \mathbf{e}_n) = (\nabla \mathbf{e}_1, \ldots, \nabla \mathbf{e}_n) = (\mathbf{e}_k \otimes \omega^k{}_1, \ldots, \mathbf{e}_k \otimes \omega^k{}_n) = (\mathbf{e}_k) \otimes (\omega^k{}_j) = \mathbf{e} \otimes \omega \,.$$

Similarly,

$$\mathrm{d}\sigma = (\mathrm{d}\sigma^1, \ldots, \mathrm{d}\sigma^n)^T = (-\omega^i{}_k \wedge \sigma^k + \tau^i) = -(\omega^i{}_k) \wedge (\sigma^k) + (\tau^i) = -\omega \wedge \sigma + \tau \,.$$

For the last equation in the section

$$\nabla \mathbf{v} = \nabla(\mathbf{e}v) = \nabla \left[(\mathbf{e}_i)(v^i)^T\right] = (\mathbf{e}_i) \otimes (\nabla v^i)^T = (\mathbf{e}_i) \otimes \left[(\mathrm{d}v^i) + (\omega^i{}_k)^T (v^k)\right] = \mathbf{e} \otimes (\mathrm{d}v + \omega v) \,.$$

9.3d. The exterior covariant differential of a vector-valued form

The notation is a bit confusing. Why $\otimes_\wedge$ and not $\wedge$? The matrix-tensor product is used in

$$\nabla \boldsymbol{\alpha} = \sum_i \mathbf{e}_i \otimes (\mathrm{d}\alpha^i + \sum_r \omega^i{}_r \wedge \alpha^r) = \mathbf{e} \otimes (\mathrm{d}\alpha + \omega \wedge \alpha) \,.$$

The coordinate-free definition is more clear.

14.3a. Tangential and normal differential forms

The book [1] defines a form α^p tangent to a submanifols (specially, to the boundary ∂M) of a compact Riemannian manifold **normal** iff $\alpha(\mathbf{T}_1, \ldots, \mathbf{T}_p) = 0$ for all vector fields $\mathbf{T}_i$ tangent to S.

A form is defined **normal** iff its Hodge dual is tangent, i.e., iff $*\alpha(\mathbf{T}_1, \ldots, \mathbf{T}_{n-p}) = 0$ for all vector fields $\mathbf{T}_i$ tangent to S. I would like to clarify the meaning of this a bit. The Hodge dual is defined as follows,

$$*\alpha = \mathrm{vol}(A, \cdots) \,,$$

which is meant as follows: A denotes the upper-index tensor corresponding to α,

$$A^{i_1,\ldots,i_p} = g^{i_1,j_1} \cdots g^{i_p,j_p} \alpha_{j1,\ldots,j_p} \,,$$

and inserting A into the volume form is the form with components

$$(\mathrm{vol}(A, \cdots))^{i_1,\ldots,i_{n-p}} = A^{j_1,\ldots,j_p} \sqrt{|g|} \epsilon_{j_1,\ldots,j_p,i_1,\ldots,i_{n-p}} .$$

This is equivalent to expressing A on the basis spanned by some basis $\mathbf{e}_1, \ldots, \mathbf{e}_k$ tangent to S, and $\mathbf{e}_{k+1}, \ldots, e_n$ transversal, and

$$*\alpha(\mathbf{v}_1, \ldots, \mathbf{v}_{n-p}) = A^{i_1,\ldots,i_p} \mathrm{vol}(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_p}, \mathbf{v}_1, \ldots, \mathbf{v}_{n-p}) \,.$$

This being normal to S means that

$$0 = *\alpha((\mathbf{T}_1, \dots, \mathbf{T}_{n-p}) = A^{i_1,\dots,i_p}\mathrm{vol}(\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_p}, \mathbf{T}_1, \dots, \mathbf{T}_{n-p})\,.$$

This means that only those components of $A^{i_1,\dots,i_p}$ may be non-zero, where $\mathbf{e}_{i_1}$, ..., $\mathbf{e}_{i_p}$ forms a linearly dependent set together with any $\mathbf{e}_{i_{p+1}}$, ..., $\mathbf{e}_{i_n}$ arbitrarily chosen basis vectors tangent to S. This always holds if $n-p>k$. Otherwise, it suffices to consider $i_1 < i_2 < \dots, i_p$, so we get that *at least* for all $i_1 > k$, $A^{i_1<i_2<\dots<i_p} = 0$. More generally, only such components can be nonzero where $i_1, \dots, i_{k-n+p} \le k$. What we see is that in the case $n-p \le k$, being tangent is a bit stronger condition then saying that $g^{-1}\alpha(\cdot, \mathbf{v}_2, \dots, \mathbf{v}_p)$ is tangent to the submanifold.

15.4a. Left-invariant fields generate right translations

To proove the title of the chapter, we need to calculate the action of $\phi_t : G \to G$ where ϕ_t is the flow of a left-invariant field $\mathbf{X}$.

The definition of a left-invariant field is that $\mathbf{X}_g = L_{g*}\mathbf{X}_e$. Let us represent the vector $\mathbf{X}_e$ with a curve g_e, such that $g_e(0) = e$, $g'_e(0) = \mathbf{X}_e$. Similarly g_g for $\mathbf{X}_g$: $g_g(0) = g$ and $g'_g(0) = \mathbf{X}_g = L_{g*}\mathbf{X}_e = \mathrm{d}gg_e(0)/\mathrm{d}t|_{t=0}$, so $g_g(t) = gg_e(t)$ is such a curve.

Considering the defintion of g_e: $g_e(0) = e$, $\mathrm{d}g_e(t)/\mathrm{d}t|_{t=0} = \mathbf{X}_e$, one such curve is $g_e(t) = \exp t\mathbf{X}_e$.

The definiton of flow is that ϕ_t is a $G \to G$ mapping for all t, and $\phi_0(g) = g$ and $\mathrm{d}\phi_0(g) = \mathbf{X}_g = \mathrm{d}gg_e(t)/\mathrm{d}t|_{t=0}$. Note that this is also satisfied by $g_g(t)$, so

$$\phi_t(g) = g_g(t) = gg_e(t) = g\exp t\mathbf{X}_e = R_{\exp t\mathbf{X}_e}(g)\,.$$

The vanishing bracket of a left-invariant field $\mathbf{X}^l$ and a right-invariant one $\mathbf{Y}_r$ follows from the expression of the bracket,

$$[\mathbf{X}^l, \mathbf{Y}^r]_g = \mathscr{L}_{\mathbf{X}}\mathbf{Y} = \lim_{t\to 0}\frac{\mathbf{Y}_{\phi_t(g)} - \phi_{t*}\mathbf{Y}_g}{t} = 0\,,$$

according to eqs. (4.4) and (4.1), and the limit vanishes, as the definition of the right-invariance of $\mathbf{Y}$ is that $\mathbf{Y}_g = R_{g*}\mathbf{Y}_e$, and so $\phi_{t*}\mathbf{Y}_g = \phi_{t*}R_{g*}\mathbf{Y}_e$, and $\phi_t(R_g(h)) = hg\exp t\mathbf{X} = R_{\phi_t(g)}(h)$, so $\phi_{t*}\mathbf{Y}_g = R_{\phi_t(g)*}\mathbf{Y}_e = \mathbf{Y}_{\phi_t(g)}$.

Do right-invariant fields generate left-translations? Consider now the curve $\tilde{g}_g(t) = g_e(t)g$. This has the properties $\tilde{g}_g(0) = g$ and $\tilde{g}'_g(0) = R_{g*}g'_e(0) = R_{g*}\mathbf{Y}_e$ if now we choose $g_e(t) = \exp t\mathbf{Y}_e$.

Let $\mathbf{Y}$ be a right-invariant vector field, $\mathbf{Y}_g = R_{g*}\mathbf{Y}_e$, and let us compare the properties of $\phi_t(g)$ where $\phi_t : G \to G$ is now the flow of the right-invariant vector field $\mathbf{Y}$, with the properties $\phi_0(g) = g$ and $\mathrm{d}\phi_t(g)/\mathrm{d}t|_{t=0} = \mathbf{Y}_g = R_{g*}\mathbf{Y}_e$. Notice that this holds for $\tilde{g}_g(t)$ too,

$$\phi_t(g) = g_e(t)g = \exp t\mathbf{Y}_e g = L_{\exp t\mathbf{Y}_e}g\,.$$

This may be used to give another proof of the vanishing of the commutator of right- and left-invariant vector fields, using Theorem (4.12),

$$
\begin{aligned}
[\mathbf{X}^l, \mathbf{Y}^r]_g f &= \frac{\mathrm{d}}{\mathrm{d}t} f(\phi^Y_{-\sqrt{t}}(\phi^X_{-\sqrt{t}}(\phi^Y_{\sqrt{t}}(\phi^X_{\sqrt{t}}(g))))) \Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} f(L_{\exp -\sqrt{t}\mathbf{Y}_e}(R_{\exp -\sqrt{t}\mathbf{X}_e}(L_{\exp \sqrt{t}\mathbf{Y}_e}(R_{\exp \sqrt{t}\mathbf{X}_e}(g))))) \Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} f(\exp(-\sqrt{t}\mathbf{Y}_e)\exp(\sqrt{t}\mathbf{Y}_e) g \exp(\sqrt{t}\mathbf{X}_e)\exp(\sqrt{t}\mathbf{X}_e)) \Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} f(g) \Big|_{t=0} = 0\,.
\end{aligned}
$$

16.3a. Connection in a vector bundle

A section of a bundle $\pi : E \to M$ is defined as follows. It is a mapping $\psi : M \to E$, $x \mapsto \psi(x)$ such that $\pi \circ \psi = \mathrm{id}_M$, i.e., $\pi(\psi(x)) = x\ p \in M$.

What is then an E valued 1-form? it is a mapping $\psi : M \times (TM)^p$ such that $\pi \circ \psi : M \times (TM)^p = \mathrm{id}_m \otimes 1$, where 1 here the constant 1 function on $(TM)^n$, i.e., for any $x \in M$, $\mathbf{v}_1, \ldots, \mathbf{v}_p \in T_x M$, $\psi_x(\mathbf{v}_1, \ldots, \mathbf{v}_p) \in \pi^{-1}(x)$.

17.1b. Principal bundles and frame bundles

This section is concerned with the definition and properties of the frame bundle. The **principal bundle** is defined as a bundle $\pi : P \to M$ where each fiber is a group G, and the transition functions act by left translation, meaning that if two local trivialisations $\pi^{-1}(U) \cong U \times G$ and $p^{-1}(V) \cong V \times G$ and $\pi(p) \in U \cap V$ (or, equivalently, $p \in \pi^{-1}(U) \cap \pi^{-1}(V)$), then

$$P \in p = \phi_U(x, g_U) = \phi_V(x, g_V)$$

and in this case there is a function $c_{VU} : G \to G$ such that $g_U = c_{UV}(x, g_V)$, in the case of a principal bundle this is of the form $g_U = c_{UV}(x) g_V$, $c_{UV}(x) \in G$.

An equivalent defintion could be given using theorem (17.8) in sec. 17.1c, that a principal G-bundle is such a bundle that the fiber is the group G, and that there is a group action

$$\hat{R} : G \times P \to P\,, \quad (g, p) \mapsto \hat{R}_g(p)$$

such that $\hat{R}_h \circ \hat{R}_h = \hat{R}_{gh}$. In the case of the principal G-bundle this action must be transitive and free (no kernel).

17.1c. Action of the structure group on a principal bundle

Some remarks for the definition of the fundamental vector field: let (P, M, π, G) be a principal G-bundle.

Theorem (17.8) could be reformulated as follows. In a local trivialisation of the bundle, $U \subset M$,

$$\pi^{-1}(U) = \Phi_U(U \times G)\,,$$

it is possible to define left and right actions of G on the bundle *locally* by $g \in G$, $L_g^{\mathrm{loc}} : G \to G$, $\Phi_U(x,h) \mapsto \Phi_U(x,gh)$ and $R_g^{\mathrm{loc}} : G \to G$, $\Phi_U(x,h) \mapsto \Phi_U(hg)$. Theorem (17.8) shows that of these, the right action can be defined *globally*, due to the commutativity of left and right action, and that the sewing functions c_{UV} act from the left.

The fundamental vector fields are defined as the tangent vectors of the curves arising from the composition of the right action of the group with a 1-parameter subgroup of G. Let $g(t) \in G$ a one-parameter subgroup, $g(0) = e$, $g(t_1 + t_2) = g(t_1)g(t_2)$. In this case, for any $p \in P$, a curve $\gamma_p(t) \in P$ can be defined as $\gamma_p(t) = pg(t)$, for which $\gamma_p(0) = p$ holds.

Let the one-parameter subgroup be $g(t) = \exp(tA)$ for an element of the Lie-algebra of G, $A \in g$. The tangent to this is the fundamental vector field, the push-forward of A through the right-action and the exponentialisation, for $\mathbf{f} = \Phi_U(x,h)$ one may define $\mathbf{f}\mathrm{e}^{tA} = \Phi_U(x, h\mathrm{e}^{tA})$ which is independent of the local trivialisation, and

$$A^*_{\mathbf{f}} := \frac{\mathrm{d}}{\mathrm{d}t} R_{\exp tA}\mathbf{f}\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{f}\exp tA\bigg|_{t=0}\,.$$

Note that as to all elements $A \in g$ corresponds a left-invariant vector field $\mathbf{A}$ on G, which has the property $\mathbf{A}_g = L_{g*}\mathbf{A}_e$, the fundamental vector field has a similar property. In a local trivialisation, there is a section of the bundle

$$\mathbf{e}_U = \Phi_U(.,e)\,,$$

where $e \in G$ is the unit element. Any point $\mathbf{f} \in P$ may be written as $\mathbf{f} = \mathbf{e}_U(\pi(f)))f_U$ where $f_U \in G$. Using these

$$A^*_{\mathbf{f}} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{f}\mathrm{e}^{tA}\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{e}_U(x)f_U\mathrm{e}^{tA}\bigg|_{t=0} = (\mathbf{e}_U(x)\cdot)_*\mathbf{A}_{f_U}\,.$$

where $x = \pi(\mathbf{f})$. This is the push-forward of $\mathbf{A}_{f_U}$ with the map $\mathbf{e}_U(x)\cdot$, and

$$\mathbf{A}_{f_U} = L_{f_U*}\mathbf{A}_e =: f_U A\,.$$

Using the more abstract formulation of the principal G-bundle (without frames), where $e_U(x) = \phi_U(x,e)$ is a unit section of the principal bundle P, it is possible to coordinatise the tangent of P by $U \times G \times g$ by assigning to x, g, A the vector

$$\frac{\mathrm{d}}{\mathrm{d}t}e_U(x)g\mathrm{e}^{tA}\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\hat{R}_{\mathrm{e}^{tA}}\hat{R}_g e_U\bigg|_{t=0}$$

at $e_U(x)g = \hat{R}_g(e_U(x))$.

18.1a. The Maurer-Cartan form

The Maurer-Cartan form is defined using a basis $\mathbf{E}_R \in g$. These are extended into a basis of left-invariant vector fields on G as

$$\mathbf{e}_{Rg} := L_{g*}\mathbf{E}_R \in T_gG\,.$$

The dual basis is denoted by σ^R,

$$\sigma^R(\mathbf{e}_S) = \delta^R_S\,, \quad \sigma^R_g(\mathbf{e}_{Rg}) = \delta^R_S\,.$$

The Maurer-Cartan g valued 1-form is defined as

$$\Omega := \mathbf{E}_R \otimes \sigma^R\,.$$

On the basis $\mathbf{e}_R$ this takes the value

$$\Omega(\mathbf{e}_R) = \mathbf{E}_S \otimes \sigma^S(\mathbf{e}_R) = \mathbf{E}_S \delta^S_R = \mathbf{E}_R\,.$$

Note that $\mathbf{e}_{Rg} = L_{g*}\mathbf{E}_R$, so

$$\Omega_g = (L_{g*})^{-1}\,.$$

In the case of a matrix group, L_g acts by matrix multiplication, so $L^{-1}_{g*} = g^{-1}$, and $\mathrm{d}g$ is the unity matrix in T_gG, so we may write

$$L^{-1}_{g*} = g^{-1}\mathrm{d}g\,.$$

Writing out $\mathrm{d}g$ is important when using a parametrisation, as $g = g(\alpha)$, so $g^*\mathrm{d}g = \partial_k g(\alpha)\mathrm{d}\alpha^k$. Usually, g^* is not written out explicitly (see the example in sec. 18.1a in the book).

The book mentions the usual proof for matrix groups that Ω is a left-invariant 1-form. Note that in the modern formulation, this is so by definition. As

$$\Omega = \mathbf{E}_R \otimes \sigma^R\,,$$

here only the σ^R's are fields, so a pull-back only acts on them, and they are left-invariant,

$$L^*_h\omega = L^*_h(\mathbf{E}_R \otimes \sigma^R) = \mathbf{E}_R \otimes L^*_h\sigma^R = \mathbf{E}_R \otimes \sigma^R = \Omega\,.$$

18.1c. Connections in a principal bundle

In the book, connections have been defined using the covariant derivative of a frame,

$$\boldsymbol{\nabla}_U \mathbf{e}_U = \mathbf{e}_U \otimes \omega_U\,,$$

where the frame $\mathbf{e}_U$ is a row-vector of vectors, and ω_U a matrix of 1-forms.

To obtain a covariant derivative of a section in the principal bundle $\mathbf{f}$, in eq. (18.12), the covariant derivative of it along a curve is defined. Let $x = x(t)$ define a curve in M, and $\mathbf{f}(t) = \mathbf{f}(x(t))$ the corresponding curve in the principal bundle. In terms of frames, $\mathbf{f}$ may be written as $\mathbf{f}(x) = \mathbf{e}_U(x)g_U(x)$, to which corresponds a curve in the group by

$$\mathbf{f}(t) = \mathbf{e}_U(x(t))g_U(x(t)) =: \mathbf{e}_U(x(t))g(t)\,.$$

We use this to obtain its covariant derivative, i.e., repeat eq. (18.12) without assuming that G is a matrix group. Points in a general principal bundle are coordinatised in a local trivialisation as

$$p(x, g) = \hat{R}_g e_U(x)\,,$$

where e_u is a unit section, and $\hat{R}$ the right-action on the bundle [see theorem (17.8)]. In this case, the covariant derivative of the curve may be defined as follows. Let $\mathbf{f}(t) = \hat{R}_{g(t)} e_U(x)$ be a curve in the bundle. In order to define

$$\frac{\boldsymbol{\nabla}\mathbf{f}}{\mathrm{d}t} = \nabla_{x'(t)}\mathbf{f} = \boldsymbol{\nabla}_{\mathbf{X}}\mathbf{f}\,,$$

where $\mathbf{X} = x'(t)$, one needs a one form taking its value in the tangent space of P. Let ω be a 1-form on M taking its value in g, the Lie algebra of G.

As $\mathbf{f}$ takes its value in the bundle P, and $\mathbf{f} = \hat{R}_g e_U$, the tangent space of P at $\mathbf{f}(t)$ may be written as $\hat{R}_{g*} T_{e_U(x)} P$, and we may identify a subset of this by g as follows: to any vector $A \in g$ we associate $\mathrm{d}(e_u(x)\mathrm{e}^{tA})/\mathrm{d}t|_{t=0}$ and we push this forward to $T_{\mathbf{f}(x)}P$ by $\hat{R}_g$, which is the value $A^*_{\mathbf{f}}(x)$ of the fundamental vector field.

This way, we may associate to $\omega(\mathbf{X}) \in g$ the vector $[\omega(\mathbf{x})]^*_{\mathbf{f}}$ in $T_{\mathbf{f}(x)}P$.

Let us now put this machinery to use. To obtain $\boldsymbol{\nabla}_{\mathbf{X}}\mathbf{f}$, where $\mathbf{f}(x) = \hat{R}_g(x) e_U(x)$ we proceed as follows.

First, we want to consider the change of $g(t) = g_U(x(t))$. The derivative of such a curve is $g'(t) \in T_g G = L_{g*} T_e G$, i.e., we may write that as $g'(t) = L_{g(t)*}\Omega(g'(t))$ where Ω is the Maurer-Cartan 1-form. We now consider the Lie algebra element $\mathrm{Ad}_{g^{-1}}(\omega_x)(\mathbf{X}) + \Omega_{g(t)}(g'(t))$ and the value of the corresponding fundamental vector field at $\mathbf{f}_U(x(t))$,

$$\boldsymbol{\nabla}_{\mathbf{X}}\mathbf{f} = \left[\left(\mathrm{Ad}_{g_U(x)^{-1}}(\omega_{g_U(x)})\right)(\mathbf{X}) + \Omega_{g_U(x)}\left(g_{Ux*}\mathbf{X}\right)\right]^*_{\mathbf{f}(x)}\,,$$

where $g_{Ux*}\mathbf{X} = \mathrm{d}g_U(x(t))/\mathrm{d}t$, and $\mathbf{X} = x'(t)$, or

$$\frac{\boldsymbol{\nabla}\mathbf{f}}{\mathrm{d}t} = \left[\left(\mathrm{Ad}_{g(t)^{-1}}(\omega_{g(t)})\right)(x'(t)) + \Omega_{g(t)}\left(g'(t)\right)\right]^*_{\mathbf{f}(t)}\,.$$

To see that this is the generalisation of eq. (18.12), we shall see that in the matrix group case $A^*_{\mathbf{f}} = g_U A$, so we get

$$\boldsymbol{\nabla}_{\mathbf{X}}\mathbf{f} = g_U\left[g^{-1}\omega(\mathbf{X})g + g^{-1}\mathrm{d}g\left(\mathbf{X}(g)\right)\right]\,,$$

and taking the frame bundle as the principal bundle, we may replace g_U before the bracket by $\mathbf{f}_U$. I think the tensor product sign in eq. (18.12) is not necessary.

18.2a. Associated bundles

The construction of the associated bundle. Let $\pi : P \to M$ be a principal bundle, and its transition functions be c, i.e., if on $U \subset M$ and $V \subset M$ two local trivialisations are given (e.g., by unit sections, e_U and e_v, then a point $P \supset \pi^{-1}(U \cap V) = e_U g_U = e_V g_V$), then $c_{UV} : U \cap V \to G$ such that $e_U = e_V c_{UV}$, and therefore $g_V = c_{VU} g_U$.

An associated bundle may be constructed as a quotient space. Let $\rho : G \to \mathrm{Gl}(X)$ be a representation of G on a vector space X. We take a chart of the principal bundle with local trivialisations, to each patch U corresponding a local unit section e_U. We want to have a bundle $\tilde{\pi} : P_\rho$such that

$$\tilde{\pi}^{-1}(U) \cong U \times X\,,$$

and the transition functions are $\tilde{c}_{UV} = \rho(c_{UV})$. What is then the total space?

We take for a chart U_α, $\cup_\alpha U_\alpha = M$ the following:

$$P_\rho = \cup_\alpha (U_\alpha \times X)/\sim$$

where the equivalence relation $\sim$ is given by

$$U \times X \ni (x, \psi_U) \sim (y, \psi_V) \in V \times X\,, \quad \text{iff } \psi_V = \rho(c_{VU})\psi_V\,.$$

The next step is the association of a bundle to a vector bundle $E \to M$ through a representation ρ of its structure group. I.e., for this bundle E, for any two patches U, V and point $x \in U \cap V$, the transition function $c_{UV}(x) \in G \subseteq GL(F)$ where the vector space F is the fiber of the bundle E.

This is constructed as follows: first, we note that the frame bundle P may be constructed with fiber G, representing sections as $\mathbf{f}(x) = \mathbf{e}_U g_U(x)$. Then choosing a representation $\rho : G \to \mathrm{Gl}(X)$ on a vector space X, the vector bundle associated to the principal frame bundle may be constructed. This will be called the bundle associated to the vector bundle through the representation ρ, i.e., $E_\rho := P_\rho$.

18.2b. Connections in associated bundles

To the use of the connection form in eq. (18.22). We have defined a connection form in a vector bundle as a matrix of 1-forms where the covariant derivative of a frame is defined as

$$\boldsymbol{\nabla}\mathbf{e}_j = \mathbf{e}_{Uk} \otimes \omega^k_{Uj}\,, \quad \text{or} \quad \boldsymbol{\nabla}\mathbf{e}_U = \mathbf{e}_U \otimes \omega_U\,.$$

Not the covariant derivative of a section $\mathbf{f} = \mathbf{e}_U f_U$ was defined in order to obey the Leibniz rule as

$$\boldsymbol{\nabla}\mathbf{f} = \boldsymbol{\nabla}(\mathbf{e}f) = \boldsymbol{\nabla}(\mathbf{e}_k f^k) = (\nabla \mathbf{e}_k) f^k + \mathbf{e}_k \mathrm{d} f^k = \mathbf{e}_k \otimes (\omega^k{}_j f^j + \mathrm{d} f^k) = \mathbf{e} \otimes (\mathrm{d} f + \omega f)\,,$$

where we have dropped the index U denoting the patch. We use the notation

$$\boldsymbol{\nabla}\mathbf{f} = \mathbf{e}_U \otimes \nabla_U f_U\,.$$

Eq. (18.22) is this, with the notation y_U for what used to be f_U.

18.3a. r-form sections of E

An r form section is defined as an anti-symmetric mapping of r vector fields to a section of a bundle E, linearly and locally, i.e., an element of

$$\Gamma(E) \otimes \bigwedge^r M\,.$$

If $\mathbf{e}_U$ is a frame (k independent local sections) of the bundle E, then an r-form can be written in the form

$$\boldsymbol{\phi}_U = \mathbf{e}_U \otimes \phi_U = \mathbf{e}_R \otimes \phi^R\,.$$

The exterior covariant derivative is defined as

$$\boldsymbol{\nabla}\boldsymbol{\phi}_U = \mathbf{e}_U \otimes (\mathrm{d}\phi_U + \omega_U \wedge \phi_U) = \mathbf{e}_R \otimes (\mathrm{d}\phi^R + \omega^R{}_S \wedge \phi^S)\,.$$

18.3b. Curvature and the *Ad* bundle

The introduction before theorem (18.40) says, a bit more explicitly, that if E is a vector bundle with transition functions c_{VU}, then the curvature forms

$$\theta_U = \mathrm{d}\omega_U + \frac{1}{2}[\omega_U, \omega_U]$$

are not the local parametrisations of a g Lie-algebra-valued 2-forms with the same structure group representation as E, i.e., not elements of $E \otimes \bigwedge^2 M$, but rather of $E_{\mathrm{Ad}} \otimes \bigwedge^2 M$.

19.3a. The Lorentz group

When considering how SO(3) is a deformation retract of L_0 the group of proper (orientation-preserving, $\{B : \det B = 1\}$) isochronous (time-direction preserving $B : B^0_0 \geq 0\{$) Lorentz transformations, there are some concepts useful in physics implicitly at play.

Let

$$H := x \in M^4 : x^i x_i = x_0^2 - \mathbf{x} \cdot \mathbf{x} = -1$$

be the hyperboloid of possible velocities of on observer.

For any $u \in H$, there is a *standard* Lorentz transformation, usually denoted $\Lambda_{u \leftarrow u_0}$, that takes the vector $u_0 = (1, \mathbf{0})^T$, into u. There are different ways to choose the standard boost.

The Lorentz tranformatios leaving a given $u \in H$ fixed,

$$L_u := \{B \in L_0 : Bu = u\}\,,$$

make up the **little group** of the given four-velocity. In the case of the standard 4-velocity $(1, \mathbf{0})^T$,

$$L_{u_0} = \begin{pmatrix} 1 & \\ 0 & \mathrm{SO}(3) \end{pmatrix}\,,$$

and for any other $u \in H$,

$$L_u = \Lambda_{u \leftarrow u_0} L_{u_0} \Lambda^{-1}_{u \leftarrow u_0}\,.$$

This is often expressed so, that *the little group corresponding to a massive momentum is* SO(3), as when choosinf a mass m, $mu = p$ is a momentum of a particle with mass m, $p^i p_i = -m^2$. It is also possible to consider the little group of a lightlike vector, which turns out to be the two-dimensional Euclidean group, ISO(2).

This yields the homogeneous space structure

$$H \cong \frac{L_0}{\mathrm{SO}(3)}\,.$$

20.4b. Averaging over a compact Lie group

In the proof of thm. (20.30), the aim is to prove that the left-invariant Haar measure (in this case, volume form), constructed from a left-invariant basis of 1-form fields,

$$\omega = \sigma^1 \wedge \cdots \wedge \sigma^n\,,$$

is bi-invariant on a compact Lie group. Here $\mathbf{e}$ is a frame of left-invariant vector fields on G, and σ is its dual.

The proof is indirect, assuming that it is not so, and then constructing a continuous function on G that diverges at some point, which contradicts compactness.

The continuous function on G is

$$F : G \to \mathbb{R}\,, \quad g \mapsto F(g) = \omega(R_{g^{-1}*}L_{g*}\mathbf{e}) = \omega(g\mathbf{e}g^{-1})\,,$$

which is well defined: the scalar field on the right has a constant value, as $L_{g*}\mathbf{e} = \mathbf{e}$, i.e., it is still a left-invariant field, and so is ω.

To show that this function diverges, it is evaluated along a sequence g, g^2, ... Assuming that $F(g)$ is not constant (i.e., that ω is not right-invarian), we may choose g such that $F(g) = c \neq F(e)$. Furthermore, we can assume that $c > 1$, otherwise replace g with g^{-1}.

As $F(g) = c = c\omega(\mathbf{e})$, it follows that $F(g^2) = c^2$, $F(g^n) = c^n$ and so $F(g^n) = c^n \to \infty$ as $n \to \infty$.

20.5a. The exterior covariant divergence ∇^*

The book gives a coordinate expression. Here, I would like to give an expression relating it to d^*.

First, note that d^* is defiend as the Hilbert space adjoint of d,

$$(\mathrm{d}\alpha, \beta) = (\alpha, \mathrm{d}^*\beta)\,.$$

This is so for ordinary forms.

Now for forms in the adjoint bundle, one needs to define the scalar product firs, which is generalised as

$$(\theta, \phi) = -\int_M \mathrm{Tr}(\theta \wedge *\phi)\,,$$

which is analogous to the formula for ordinary forms, except that here, the exterior product takes its value in the tensor product of the Ad bundle with itself. As any bilinear function can be extended to the tensor product, so can the $-\mathrm{Tr}$.

Let us now consider $\nabla\theta$, which is

$$\nabla\psi = \mathrm{d}\psi + \mathrm{Ad}_*(\omega)\psi = \mathrm{d}\psi + [\omega, \psi]\,.$$

Note, that we do know the adjoint of d, its d^*, so

$$(\nabla\psi, \phi) = (\mathrm{d}\psi + [\omega, \psi], \phi) = (\mathrm{d}\psi, \phi) + ([\omega, \psi], \phi) = (\psi, \mathrm{d}^*\phi) + ([\omega, \psi], \phi)\,.$$

Let us consider the second term,

$$([\omega, \psi], \phi) = -\int_M \mathrm{Tr}([\omega, \psi] \wedge *\phi) = \int_M \mathrm{Tr}(\psi[\omega, \phi]) = -(\psi, [\omega, \phi])\,,$$

where we are using eq. (20.35) in the form $\mathrm{Tr}([X, Y], Z) = -\mathrm{Tr}(Y, [X, Z])$, so we arrive at

$$\nabla^*\phi = \mathrm{d}^*\phi - [\omega, \phi]\,.$$

21.1a. Bi-invariant p-forms

The end of the proof,

$$\alpha^p = \alpha_{\underset{\to}{I}} \sigma^I = \alpha_{\underset{\to}{I}} \tau^I \,,$$

where the σ^i are left- and the τ^i are right-invariant forms, which agree at the identity, $\sigma^i_e = \tau^i_e$.

Accoding to sec. 15.4c,

$$\mathrm{d}\sigma^i = -\frac{1}{2} C^i_{jk} \sigma^j \wedge \sigma^k \,, \quad \mathrm{d}\tau^i = \frac{1}{2} C^i_{jk} \tau^i \wedge \tau^k \,,$$

where C^i_{jk} are the structure constants of the group. On one hand,

$$\mathrm{d}\alpha = -\frac{1}{2} \alpha_{i_1 < \cdots < i_p} \left(C^{i_1}_{jk} \sigma^j \wedge \sigma^k \wedge \sigma^{i_2} \wedge \cdots \wedge \sigma^{i_p} - C^{i_2}_{jk} \sigma^{i_1} \wedge \sigma^j \wedge \sigma^k \wedge \sigma^{i_3} \wedge \cdots \wedge \sigma^{i_p} + \ldots \right) \,,$$

and on the other,

$$\mathrm{d}\alpha = +\frac{1}{2} \alpha_{i_1 < \cdots < i_p} \left(C^{i_1}_{jk} \tau^j \wedge \tau^k \wedge \tau^{i_2} \wedge \cdots \wedge \tau^{i_p} - C^{i_2}_{jk} \tau^{i_1} \wedge \tau^j \wedge \tau^k \wedge \tau^{i_3} \wedge \cdots \wedge \tau^{i_p} + \ldots \right) \,.$$

Note, that the two formulae only differ in the sign, and the replacement $\sigma \mapsto \tau$, so evaluating at $g = e$,

$$\mathrm{d}\sigma_e = -\mathrm{d}\sigma_e \,, \quad \mathrm{d}\sigma_e = 0 \,,$$

and as σ is invariant, so is $\mathrm{d}\sigma$, so $\mathrm{d}\sigma = 0$.

A.c. Symmetry of Cauchy's stress tensor in $\mathbb{R}^3$

The logic here is as follows: the angular momentum is defined first (5 eqs. before A.11), as

$$-H = \frac{1}{2} \int_{B(t)} \mathbf{r} \wedge \mathbf{v} \otimes m \,,$$

i.e., $-\mathbf{r} \wedge \mathbf{v} \otimes m$ is defined as the angular momentum density. Its derivative is then obtained (i) *directly*, by inserting Cauchy's equations (A.10) into $\mathrm{d}H/\mathrm{d}t$, and (ii) *with the assumption*, that it agrees with all torques acting on the body. The condition that the two agree is

$$\int_{B(t)} \mathrm{d}\mathbf{r} \wedge \mathbf{t} = 0 \,,$$

which is equivalent to the symmetry of Cauchy's stress tensor, or

$$\mathrm{d}x^r \wedge t^s = \mathrm{d}x^s \wedge t^r \,.$$

E.a. The topology of conjugacy orbits

Let us consider the mapping F. It is defined as the mapping of the cosets of G/C_σ to $M_\sigma \subset G$ as follows,

$$F : G/C_\sigma \to M_\sigma \subset G, \quad gC_\sigma \mapsto g\sigma g^{-1}.$$

First, let us show that this map is well-defined, i.e., if g and g' are in the same coset, $g' = gh$, $h \in C_\sigma$, then $g'\sigma g'^{-1} = gh\sigma(gh)^{-1} = gh\sigma h^{-1}g^{-1} = g\sigma hh^{-1}g^{-1} = ghg^{-1}$.

Secondly, to show that F is an embedding of the manifold G/C_σ in M_σ, it is necessary to show that F_* is 1:1. This is first fone at $\sigma C_\sigma \in G/C_\sigma$,

$$F_{*\sigma C_\sigma} : T_{\sigma C_\sigma}(G/C_\sigma) \to T_\sigma M_\sigma \subset T_\sigma G\,,$$

using the fact that $\sigma C_\sigma = C_\sigma$ and that curves in G/C_σ can be parametrised as $g(t)C_\sigma$ and at σC_σ, these curves can be taken to be of the form $\mathrm{e}^{Yt}C_\sigma$, which are mapped into $\mathrm{e}^{Yt}\sigma\mathrm{e}^{-Yt}$.

The velocity vector of $\mathrm{e}^{Yt}\sigma\mathrm{e}^{-Yt}$ is $R_\sigma Y - L_\sigma Y$, and if this vanishes, then $L_{\sigma^{-1}}R_\sigma Y = \mathrm{Ad}_\sigma Y = Y$, as a consequence, e^{Yt} and σ commute, the curve is on C_σ, i.e., it is a constant in G/C_σ.

As $\mathrm{Ad}_g : G \to G$ is a diffeomorphism, it is clear that F_* is 1:1 everywhere, so F is a local embedding everywhere. It can be shown that M_σ is globally an embedded submanifold.

Our calculation also shows that the mapping F_{*C_σ} mapping $Y \mapsto R_\sigma Y - L_\sigma Y$ is $1:1$ and onto.

References

[1] Theodore Frankel, *"The geometry of physics"* (Cambridge University Press, Cambridge, UK, 1997).